# Conference
# Geometric and Topological Properties of Random Algebraic Varieties

4 – 6 October 2023, Universität zu Köln

| | Wednesday 4 Oct | Thursday 5 Oct<br>*Day in honour of Prof. Dinh's birthday* | Friday 6 Oct |
|---|---|---|---|
| 8:00–9:00 | **Registration** | | |
| 9:00–9:50 | **Welschinger** | **Coman** | **Dinh** |
| 10:00–10/50 | **Černý** | **Gayet** | **Vidotto** |
| 11:00–11:30 | Coffee | Coffee | Coffee |
| 11:30–12:20 | **Kabluchko** | **Bayraktar** | **Hirsch** |
| 12:30–14:00 | Lunch | Lunch | Lunch |
| 14:00–14:50 | **Wigman** | **Nguyên** | Departure |
| 15:00–15:50 | **Winter** | **Ancona** | |
| 16:00–16:30 | Coffee | Coffee | |
| 16:30–17:20 | **Puchol** | **Ghosh** | |
| | Wine & Cheese | Conference dinner | |

## Talk titles and abstracts:

**Michele Ancona (Nice)**

**Title:** Metric and spectral aspects of random plane curves
**Abstract:** A (complex) plane curve is the zero locus in $\mathbb{CP}^2$ of a homogeneous complex polynomial in three variables. Any plane curve is endowed with a Riemannian metric induced by the ambient Fubini–Study metric of the complex projective plane. We give probabilistic lower bounds on some metric and spectral quantities (such as the systole or the spectral gap) of the plane curves when these are chosen randomly in the Fubini–Study ensemble. This is a joint work with Damien Gayet.

**Turgay Bayraktar (Istanbul)**

**Title:** Mass equidistribution for random holomorphic sections
**Abstract:** In the first part of this talk, I will review some basic results on the Quantum Unique Ergodicity conjecture concerning the zero distribution of large frequency eigenfunctions of the Laplacian on a negatively curved manifold. By replacing Laplace eigenfunctions with global holomorphic sections associated with a line bundle endowed with a singular Hermitian metric, one is led to their geometric counterparts. In the second part, I will discuss some recent results on limiting mass distribution of random holomorphic sections and random orthonormal bases.

**Jiří Černý (Basel)**

**Title:** Complexity of spin–glass Hamiltonians
**Abstract:** I review certain results on the complexity and topology of the energy landscapes of spherical spin glasses, and their connection to random matrices. These energy landscapes can be viewed as random polynomials on a high–dimensional sphere. The distribution of their critical points and topology should play an important role in rigorous understanding of the physics of spin glasses. In the end I present new concentration results for these objects.

**Dan Coman (Syracuse)**

**Title:** Restricted spaces of holomorphic sections vanishing along subvarieties
**Abstract:** Let $L$ be a holomorphic line bundle on a compact normal complex space $X$ of dimension $n$, let $\Sigma = (\Sigma_1, \dots, \Sigma_\ell)$ be an $\ell$–tuple of distinct irreducible proper analytic subsets of $X$, and $\tau = (\tau_1, \dots, \tau_\ell)$ be an $\ell$–tuple of positive real numbers. We consider the space $H^0_0(X, L^p)$ of global holomorphic sections of $L^p := L^{\otimes p}$ that vanish to order at least $\tau_j p$ along $\Sigma_j$, $1 \le j \le \ell$, and give necessary and sufficient conditions to ensure that $\dim H^0_0(X, L^p) \sim p^n$.

If $Y \subset X$ is an irreducible analytic subset of dimension $m$, we also consider the space $H^0_0(X|Y, L^p)$ of holomorphic sections of $L^p|_Y$ that extend to global holomorphic sections in $H^0_0(X, L^p)$, and we give a general condition on $Y$ to ensure that $\dim H^0_0(X|Y, L^p) \sim p^m$. When $L$ is endowed with a continuous Hermitian metric, we show that the Fubini–Study currents of the spaces $H^0_0(X|Y, L^p)$ converge to a certain equilibrium current on $Y$, and we apply this to the study of the equidistribution of zeros in $Y$ of random holomorphic sections in $H^0_0(X|Y, L^p)$ as $p \to \infty$.

This is joint work with George Marinescu and Viêt–Anh Nguyên.

**Tien–Cuong Dinh (Singapore)**

**Title:** Hole event for holomorphic sections on Riemann surfaces
**Abstract:** Let $L$ be positive line bundle over a compact Riemann surface. We consider the zeros of a random holomorphic section of $L^n$ conditioned by the hole event that there are no zeros in a given open set $D$. We prove the existence of a forbidden region and the equidistribution of the zeros outside the set $D$ and the forbidden region when $n$ goes to infinity. This is a joint work with Subhro Ghosh and Wu Hao.

**Damien Gayet (Grenoble)**

**Title:** Local topology of random complex algebraic hypersurfaces
**Abstract:** Projective complex algebraic smooth hypersurfaces of a given degree have the same (global) topology, and the Lefschetz hyperplane theorem asserts that only the middle homology group is non trivial. I will explain that these two miraculous properties have some local echo for random hypersurfaces.

**Subhro Ghosh (Singapore)**

**Title:** Rigidity phenomena in strongly correlated random point fields and the emergence of forbidden regions
**Abstract:** The aim of this talk is to give an introduction to recent results concerning rigidity properties of random point fields, focussing in particular on the associated phenomenon of forbidden regions. Rigidity phenomena pertain to the emergence of singularities in strongly correlated point processes (e.g. those related to random matrices, Coulomb gases, zeros of random polynomials and random sections of line bundles), wherein local conservation laws have been shown to hold true. This includes, e.g., local conservation laws for mass or centre of mass, conditioned on the environment (ie the configuration of points outside a given domain). Such rigidity phenomena have been shown to be closely related to the emergence of novel `forbidden regions' in the spatial distribution of points conditioned on important classes of rare events. In particular, spatial holes (i.e. vacant domains) of macroscopic size (at the scale of the droplet) in strongly rigid point fields have been shown to enforce the emergence of complementary holes (of comparable scale) in other parts of the space. Based in part on the following joint works.
[1] Rigidity and Tolerance in point processes: Gaussian zeroes and Ginibre eigenvalues, with Y. Peres, Duke Mathematical Journal, 166 (10), 1789–1858.
[2] Gaussian complex zeros on the hole event: the emergence of a forbidden region, with A. Nishry, Communications in Pure Appl. Math. (CPAM), 72, no. 1 : 3–62.

[3] Approximate Gibbsian structure in strongly correlated point fields and generalized Gaussian zero ensembles, with U. Gangopadhyay, K.A. Tan, Communications in Pure Appl. Math. (CPAM), to appear.
[4] Forbidden regions for random zeros on Riemann surfaces, with T.C. Dinh, H. Wu (near completion).

**Christian Hirsch (Aarhus)**

**Title:** Functional central limit theorems for Betti numbers of Gaussian excursion sets
**Abstract:** Let $F(t)$, $t \in \mathbb{R}^d$ be a sufficiently smooth stationary Gaussian field. Then, the excursion set $A(u) := \{t \in \mathbb{R}^d : F(t) \geq u\}$ above a level $u$ can be thought of as a random manifold with boundary whose geometry is important in many applications. In this talk, I will concentrate on the Betti numbers of the excursion set in growing sampling windows $W_n \subset \mathbb{R}^d$. I will present my joint ongoing work with R. Lachièze–Rey (Paris Cité) to proving a central limit theorem (CLT) for these Betti numbers in the sub–critical regime of excursion–set percolation.

We are inspired by work of Beliaev, McAuley, and Muirhead on the component count, which does not have any percolation restrictions. As a key ingredient, we rely on a recent result by Gass and Stecconi on the finiteness of moments of critical points. Moreover, an important property of our CLT is that it is functional in the level $u$.

**Zakhar Kabluchko (Münster)**

**Title:** Zeros of random polynomials undergoing the heat flow
**Abstract:** We investigate the evolution of the empirical distribution of the complex roots of high–degree random polynomials, when the polynomial undergoes the heat flow. In one prominent example of Weyl polynomials, the limiting zero distribution evolves from the circular law into the elliptic law until it collapses to the Wigner semicircle law, as was recently conjectured for characteristic polynomials of random matrices by Hall and Ho, 2022. Moreover, for a general family of random polynomials with independent coefficients and isotropic limiting distribution of zeros, we determine the zero distribution of the heat–evolved polynomials in terms of its logarithmic potential. Furthermore, we explicitly identify two critical time thresholds, at which singularities develop and at which the limiting distribution collapses to the semicircle law. Under mild conditions, we provide a complete characterization of the limiting distribution of heat–evolved random polynomials as a push–forward of the initial distribution under a transport map. Finally, we discuss the results from the perspectives of (partial) differential equations (in particular Hamilton––Jacobi equation and Burgers' equation), free probability and optimal transport. The theory is accompanied by explicit examples, simulations and conjectures. The talk is based on

a joint work with Brian Hall, Ching–Wei Ho and Jonas Jalowy: https://arxiv.org/abs/2308.11685, https://arxiv.org/abs/2304.06665.

**Viêt–Anh Nguyên (Lille)**

**Title:** The generalized Lelong numbers and intersection theory
**Abstract:** The notion of Lelong number $\nu(T,x)$ of a positive closed current $T$ at a single point $x$ in an ambient complex manifold $X$ plays a fundamental role in Complex Analysis and Complex Geometry. In 1982 Henri Skoda formulated this notion for the more general class of positive plurisubharmonic currents. In this talk we introduce a new concept of the generalized Lelong numbers $\nu_j(T,V)$, where $V$ is a submanifold in $X$ and $T$ is a positive plurisubharmonic current in $X$. In general, we have $\dim V+1$ generalized Lelong numbers associated to $T$ along $V$. The classical case where $V=\{x\}$ corresponds to $\dim V=0$.

Our present research is inspired by two works. The first one is the theory of tangent currents for positive closed currents which were developed by Tien–Cuong Dinh and Nessim Sibony (2018). The second work is the theory of the Lelong number for positive plurisubharmonic currents along a complex linear subspace in $\mathbb{C}^n$ which were developed by Lucia Alessandrini and Giovanni Bassanelli (1996).

Next, we study these new numerical values and establish their basic properties. In particular, we obtain geometric characterizations as well as an upper–semicontinuity of the generalized Lelong numbers in the sense of Yum–Tong Siu (1974). When the current $T$ is positive closed, we also establish some links between the generalized Lelong numbers and Dinh–Sibony cohomology classes. Finally, as an application we give an effective condition (in terms of the generalized Lelong numbers) ensuring that $m$ positive closed currents $T_1,\dots,T_m$ of possibly different bidegrees $(p_j,p_j)$ for $1\leq j\leq m$ on a compact Kähler manifold $X$ are wedgeable in the sense of Dinh–Sibony.

**Martin Puchol (Orsay)**

**Title:** Variance of the volume of random real algebraic submanifolds
**Abstract:** The object of interest in this talk is the common zero set $Z_d$ of $r$ independent random polynomials of degree $d$ in the real $n$–dimensional projective space. We will study the asymptotics of the variance of its volume as the degree grows to infinity, and give some applications, such as asymptotic a.s. density. This study will be in fact carried out in the more general setting of real projective manifold endowed with growing tensor powers of a real ample line bundle.

**Anna Vidotto (Naples)**

**Title:** Functional Convergence of Berry's Nodal Lengths and Total Disorder
**Abstract:** In this talk, we consider Berry's random planar wave model (1977), and present spatial functional limit theorems – in the high–energy limit – for discretized versions of the random field obtained by restricting its nodal length to rectangular domains. We will see that proving our limit theorems require a detailed study of the projection of nodal lengths onto the so–called second Wiener chaos. Such study leads to a result of independent interest: the high–energy fluctuations of the nodal length's second chaotic components are given by a Gaussian total disorder field indexed by polygonal curves – how we obtain this total disorder process in the limit will be shown in detail.

The talk is based on a joint work with M. Notarnicola and G. Peccati.

**Jean–Yves Welschinger (Lyon)**

**Title:** Expected area of the amœba of a random plane curve
**Abstract:** I will discuss a joint work with Özgür Kişisel whose aim is to estimate the area of the amoeba of a random plane curve.

**Igor Wigman (London)**

**Title:** Almost sure GOE fluctuations of energy levels for hyperbolic surfaces of high genus
**Abstract:** This talk is based on a joint work with Zeev Rudnick.
We study the variance of a linear statistic of the Laplace eigenvalues on a hyperbolic surface, when the surface varies over the moduli space of all surfaces of fixed genus, sampled at random according to the Weil–Petersson measure. The ensemble variance of the linear statistic was recently shown to coincide with that of the corresponding statistic in the Gaussian Orthogonal Ensemble (GOE) of random matrix theory, in the double limit of first taking large genus and then shrinking size of the energy window. We show that in this same limit, the energy variance for a typical surface is close to the GOE result, a feature called "ergodicity" in the random matrix theory literature.

**Anita Winter (Essen)**

**Title:** Spaces of algebraic measure trees and triangulations of the circle
**Abstract:** In this talk we introduce algebraic trees which can be considered as (continuum) metric trees that focus on the tree structure rather than on metric distances. We further equip algebraic trees with a sampling (probability) measure. This measure gives rise to the branch point distribution which turns out to be the

length measure of an intrinsic choice of a metric tree representation. We will provide a notion of convergence of algebraic measure trees which resembles the idea of the Gromov–weak topology which itself is defined through weak convergence of sample distance matrices. Binary algebraic (measure) trees are of particular interest because they often arise in praxis, and also due to their close connection to triangulations of the circle. We use this connection to show that in the subspace of binary algebraic measure trees, weak convergence of sample shapes, sample subtree masses and sample distance matrices are all equivalent and define a compact, metrizable topology. Furthermore, the coding by triangulations is a continuous, surjective operation in this topology. (this is joint work with Wolfgang Löhr).